Motion of a Symmetric Top Analyzed by Circular Motion

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ABSTRACT

When a symmetrical rigid top rotates on a plane, it will encounter resistance to prevent it from falling due to gravity. According to various theoretical inferences and practical proofs, the original vertical kinetic energy of the top is converted into horizontal kinetic energy because of unknown reason. I attribute this unknown reason to the horizontal circular motion of the top, and derive the specific movement trajectory of the top base on this, including three movement trajectories and their conditions.

Function Between Angle of Circle and Height

Before we dive into the analysis, there is an important step, which is to derive the function between the angle and the height of a circle, and we will use this function in the subsequent analysis.

As Fig.1, under the action of gravity, a non-rotating top standing on a fixed point will fall along a circle with this fixed point as the center. Starting from the initial angle relative to the Z-axis, top rotates through an angle $\Delta\theta$ while descending a distance h.

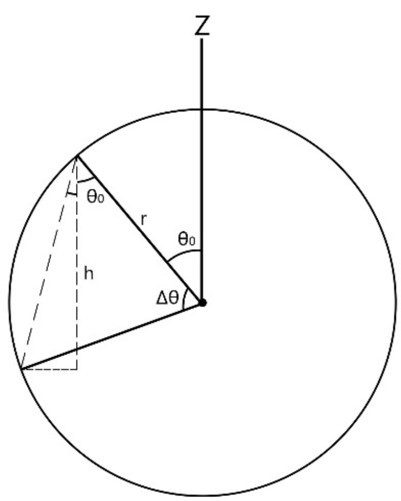

Figure 1

We can get a formula:

$$h = 2r * \sin\left(\frac{1}{2}\Delta\theta\right) * \cos\left(\frac{\pi-\Delta\theta}{2} - \theta_0\right) \quad \text{(Equ.1)}$$

Convert Equ.1, we can obtain:

$$\Delta\theta = \sin^{-1}\left[\frac{h}{r} - \sin\left(\frac{\pi}{2} - \theta_0\right)\right] + \frac{\pi}{2} - \theta_0 \quad \text{(Equ.2)}$$

Using Circular Motion to Analyze the Motion of a Symmetric Top

It turns out that the top will do circular motion rather than falling when its rotation speed around its own axis of symmetry is high enough, which means there is an unknown force prevents the top from falling down and provides the centripetal force required for circular motion.

The motion of the top can be described by Euler's angles φ, θ, and ψ, as shown in Fig.2 and Fig.3. The inertia space is defined by the system as X, Y, Z, and the origin O is located at the bottom of the top. Since the gravitational field is uniform, the effect of gravity can be described as the resultant of gravitational forces acting on the center of gravity CG of the top, which has a distance d from the bottom of the top along the symmetry axis z. Additionally, the bottom of the top is connected to the origin O by a rod of length r with negligible mass.

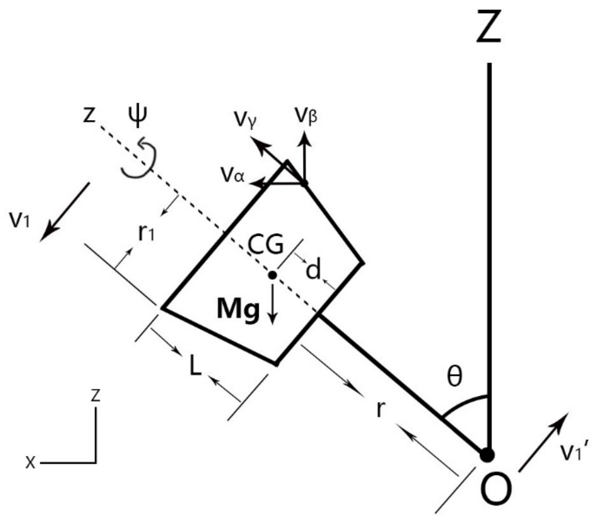

Figure 2

First of all, according to the Fig.2, it can be imagined that if there is not atmospheric friction and no Z-axis rod, the spinning top will fall down due to the gravity regardless of whether it rotates around the z-axis, and the relative velocity of any points of the top are the same from the view of x-z plane. In the other words, there is not speed difference between any points on the top from the view of x-z plane, including the top and the apex of the top. But if the apex of the top is connected via the Z-axis rod, the velocity of the apex must be zero relative to the rod. But the other parts of the top should fall due to the gravity and get a resultant velocity V_1 at the CG of the top, which results in the apex having the same magnitude but opposite direction velocity, V_1', respect to the CG of the top. In addition, V_1' is generated by the support force of the Z-axis rod.

Since the apex has the velocity V_1' relative the CG of the top, this causes the top to rotate around the CG in the direction of the linear velocity V_1', and there is a velocity V_γ at a certain point of the top side of the spinning top. Let's first focus on the top cross-sectional area of the top, in this assumption, V_γ is located at the intersection of the top cross-sectional area and the top edge.

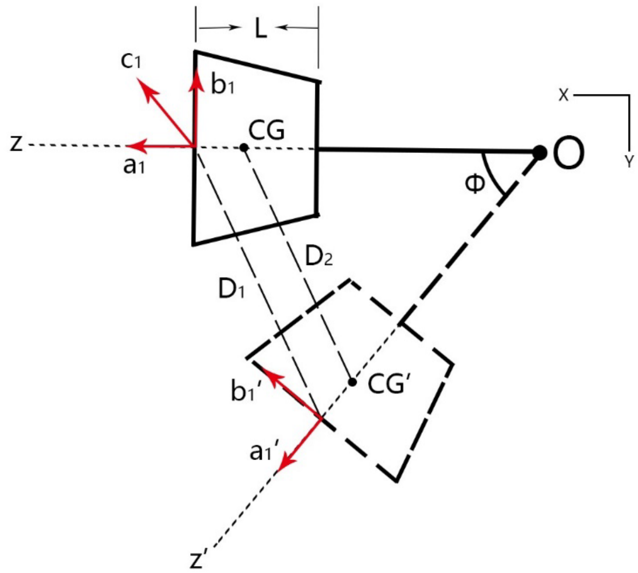

Figure 3

It's easier to deduce if using θ = 90° and look at the system from a top view, as shown in Fig.3. In this case, $b_1 = \psi * r_1$, $a_1 = V_\gamma$. Since the top cross-sectional area is subjected to these two velocities, it is supposed to obtain a resultant velocity c_1 with different direction and higher magnitude, as shown in Fig.3. But it turns out the rotation speed of the top around the z-axis is constant during the precession and nutation when the system is in the absence of other work, that means c_1 has the same magnitude as b_1. In order to ensure that c_1 and the rotation direction around the z-axis are the same, the top must move from the original location described by solid line in Fig.3 to the new location described by dashed line, and the moving distance is D_1.

Then, the similar triangle theorem can be used in the case that only the top cross-sectional area of the top is focused:

$$\frac{b_1}{a_1} = \frac{r+L}{D_1} \quad \text{(Equ.3)}$$

$$\frac{\psi * r_1}{V_\gamma} = \frac{r+L}{(r+L)\phi} \quad \text{(Equ.4)}$$

Now, Let's add some constrains, θ is an arbitrary angle, limited to 0 ≤ θ ≤ π, instead of fixed value 90°; The moving distance of CG, D_2, will be considered rather than D_1; The top cross-sectional area will be still focused. Furthermore, the circular motion of the top that parallel to xy-plane will be considered, so V_α is used instead of V_γ. Finally, Equ.5 will be obtained according to Equ.4.

$$\frac{\psi * r_1}{V_\alpha} = \frac{r+d}{(r+d)\phi} \quad \text{(Equ.5)}$$

Even if the different θ results in the different relative position of C_1 and Z-axis rod, the similar triangle theorem can still be used. Just a reminder, the vector b_1 always lies in the xy-plane. Then, Equ.5 can be derived as follows.

$$\frac{\psi * r_1}{\frac{V_{1(top)}{}' r_1}{r+l} * \sin(\theta)} = \frac{r+d}{(r+d)\phi} \quad \text{(Equ.6)}$$

$$\phi = \frac{V_1{}' * \sin(\theta)}{\psi(r+l)} \quad \text{(Equ.7)}$$

Then multiply both sides of Equ.7 by (r+d)sin(θ):

$$D_2 = \frac{{V_{1(top)}}'}{\psi(r+l)} * (r+d)sin^2(\theta) \qquad \text{(Equ.8)}$$

Since the top will oscillate up and down, so considering the elapse of infinitesimal time, the difference of D_2 will be dD_2, and the difference of $V_{1(top)}$ will be $dV_{1(top)}$. Then, divide both sides of Equ.8 by dt:

$$\dot{D}_2 = \frac{{\dot{V}_{1(top)}}'}{\psi(r+l)} * (r+d)sin^2(\theta) \qquad \text{(Equ.9)}$$

Because $V_{1(top)}$' must be equal to $V_{1(top)}$, so ${\dot{V}_{1(top)}}'$ is also equal to $\dot{V}_{1(top)}$. Then, replace $\dot{D}_2$ and ${\dot{V}_{1(top)}}'$ with V_2 and $a_{1(top)}$ respectively:

$$V_2 = \frac{a_{1(top)}}{\psi(r+l)} * (r+d)sin^2(\theta) \qquad \text{(Equ.10)}$$

$$a_{1(top)} = \frac{\psi\, V_2}{(r+d)sin^2(\theta)}(r+l) \qquad \text{(Equ.11)}$$

Equ.11 is the equation about $a_{1(top)}$ and V_2 for the top cross-sectional area of the top, even though we can choose any value for l. So the integration is needed to get the equation for the entire top:

$$a_1 = \int_0^l a_{1(top)} = \frac{\psi\, V_2}{(r+d)sin^2(\theta)}\int_0^l (r+l)dl \qquad \text{(Equ.12)}$$

$$a_1 = \frac{\psi\, V_2\, l(r+0.5l)}{(r+d)sin^2(\theta)} \qquad \text{(Equ.13)}$$

Then, the kinetic energy in the falling direction of the top falling from rest can be got:

$$0.5m{V_1}^2 = m\int(g * \sin(\theta) - a_1)ds \qquad \text{(Equ.14)}$$

In Equ.14, s is the real distance the top moves, θ is the function related to s or h, and V_1 is the velocity in the falling direction. But for better calculation, the equation likes ($0.5mv^2$=mgh) will be used instead of Equ.14. In the other words, $ds = dh/\sin(\theta)$, so Equ.14 will be converted:

$$0.5m{V_1}^2 = m\int\left(g - \frac{\psi\, V_2\, l(r+0.5l)}{(r+d)sin^3(\theta(h))}\right)dh \qquad \text{(Equ.15)}$$

Where h is the vertical height that the top passes.

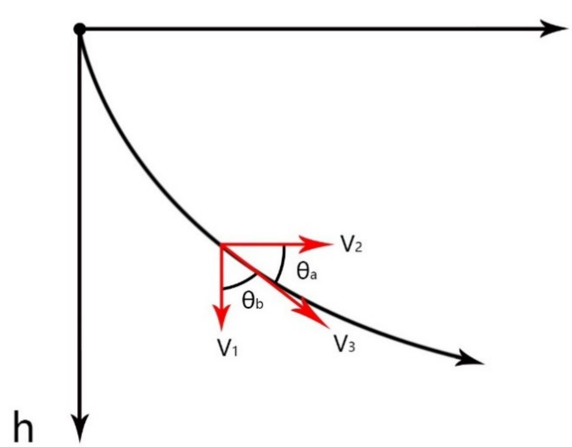

Figure 4

Fig.4 above shows the trajectory of the top falling from rest. As shown in Fig.4, $V_2 = \cos(\theta_a)\sqrt{2gh}$. Furthermore, ${V_3}^2 = 2gh$, but for ${V_1}^2$, g is a function $g * sin^2(\theta_a)$ instead of constant, so ${V_1}^2 = 2\int(g * sin^2(\theta_a))dh$

Therefore, Equ.15 can be replaced and simplified, and another thing is that θ(h) can be replaced by $\Delta\theta + \theta_0$:

$$0.5m * 2\int(g * sin^2(\theta_a))dh = m\int\left(g - \frac{\psi\, l(r+0.5l)\cos(\theta_a)\sqrt{2gh}}{(r+d)sin^3(\theta(h))}\right)dh \quad \text{(Equ.16)}$$

$$\cos(\theta_a) = \frac{\psi\, l(r+0.5l)\sqrt{2gh}}{g(r+d)sin^3(\Delta\theta+\theta_0)} \quad \text{(Equ.17)}$$

Substituting Equ.2 into Equ.17 and simplifying:

$$\theta_a = cos^{-1}\left(\frac{\psi\, l(r+0.5l)\sqrt{2gh}}{g(d+r)\left\{1-\left[\frac{h}{d+r}-\cos(\theta_0)\right]^2\right\}^{3/2}}\right) \quad \text{(Equ.18)}$$

or

$$\theta_b = sin^{-1}\left(\frac{\psi\, l(r+0.5l)\sqrt{2gh}}{g(d+r)\left\{1-\left[\frac{h}{d+r}-\cos(\theta_0)\right]^2\right\}^{3/2}}\right) \quad \text{(Equ.19)}$$

The graph drawn by Equ.18 has three trajectories of θ_α basically, and it was found that the initial angle θ_0 has a great influence on the trajectory of the top. In addition, θ_0 can be assumed from 0 to 0.5π:

1. If θ_0 is relatively large, a graph similar to a steep hill will appear. This case is shown in Fig.5(a). Also, this case will appear if θ_0 is larger than 0.5 π.
2. When θ_0 is relatively medium, θ_α will first drop sharply and then rise suddenly, forming a sharp point, and then drop to zero again. As shown in Fig.5(b).
3. If θ_0 is relatively small, θ_α will also decrease sharply first until it reaches the horizontal axis. The graph rises again after a certain distance and then drops to zero again, forming a U-like shape.

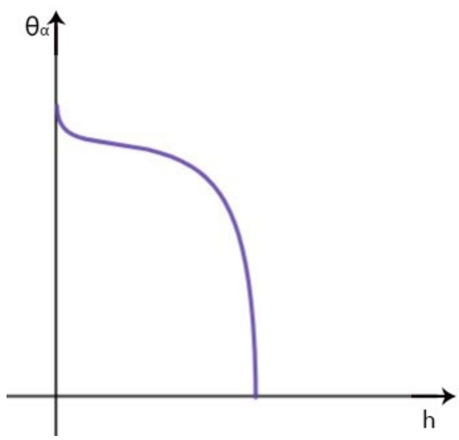

Fig.5(a)

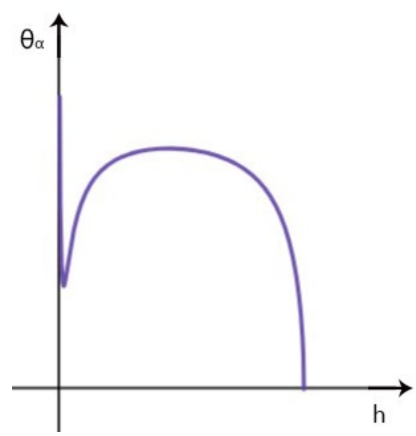

Fig.5(b)

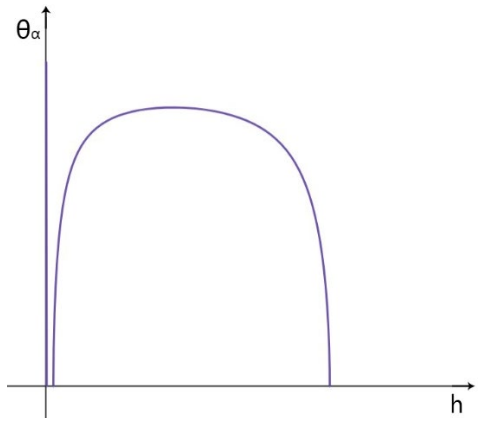

Fig.5(c)

Of course, other initial conditions also have an influence on the trajectory of the top. And the reason I chose Equ.18 to create Fig.5 is that it is more intuitive.

Since θ_α has three trajectories, the top also has three distinct cases. The motion orbit of the top can be generated by integrating tan(Equ.19) on the website Desmos Graphing Calculator. Furthermore, Fig.5(a), Fig.5(b), and Fig.5(c) correspond to Fig.6(a), Fig.6(b) and Fig.6(c), respectively.

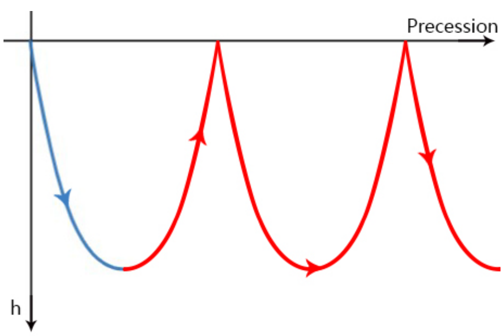

Fig.6(a)

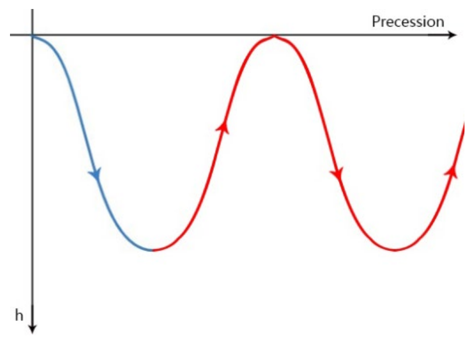

Fig.6(b)

The blue trajectories of Fig.6 are generated by Desmos website, because the laws of movement are not affected by space and time in the inertia reference frame, so the subsequent orbits (the red trajectories) can be inferred.

Since Fig.5(c) exceeds the limit from a mathematical standpoint, so the Desmos website cannot generate the integration in this case. But it can be seen from Fig.5(c) that θ_α decreases sharply from 0.5 π to 0, and there is still a decreasing trend. This means that the top will spin in a small circle to return to the starting point. That is to say, from the perspective of coordinate system in Fig.6, the top will spin in a very small circle and return to the origin, and continue to move in the negative direction of x-axis (or Precession axis) and downward. After a small distance on h-axis in Fig.5(c), the angle appears again, but it turns out that it is impossible that the angle reappears as 0 degree. Therefore, to obtain the same value of the cos function, θ_α can be 0 or $\pm 2\pi$. Since θ_α starts at $\pi/2$ and is initially downward in the negative direction, so it can be assumed that θ_α reappears as -2π, that means θ_α just finishes a circle and the true angle of θ_α at that point is $\pi/2-2\pi$, or $\pi/2$.

After θ_α reappears, it continues to rise, that is, θ_α becomes $\pi/2-(2\pi+x)$, which eventually causes θ_α to become smaller. After the graph line of Fig.5(c) reaches its vertex, it will drop again and reaches 0 degree. If this part still represents $\pi/2-(2\pi+x)$, then in the end it will become $\pi/2$ again, which is incorrect. So the part of the graph line of Fig.5(c) after it touches the vertex should represent the true angle of θ_α, Instead of $\pi/2-(2\pi+x)$.

Based on the above analysis, the following figure can be obtained.

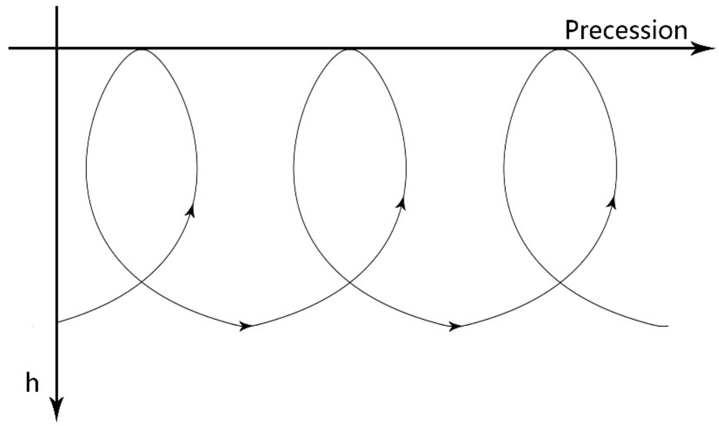

Fig.6(c)

In addition to the initial angle θ_0, the angular speed ψ also has a great influence on the trajectories of the top:

1. From Fig.6(a) generated from the Desmos website, it can be seen that for the blue part of a-trajectory (the trajectory shown in Fig.6(a)), the higher the ψ, the smaller the range of h-axis and Precession-axis, which indicates that the amplitudes of the nutation and precession become smaller with increasing ψ.
2. However, the b-trajectory has a different case compared with a-trajectory, From the Fig.5(b) and Fig.6(b) generated from the Desmos website, it can be seen that as ψ increases, the range of h-axis becomes smaller, but the range of Procession-axis becomes larger. Furthermore, b-trajectory has a tendency to transform into the c-trajectory.
3. The c-trajectory is special as well. From Fig.5(c) generated by the Desmos website, it can be seen that as ψ is increased, the range of h-axis and the size of the semicircle-like shape become increasingly small, and the interval between the origin and the semicircle becomes larger as well. Until the semicircle disappears. It seems that c-trajectory will jump into the steady precession under certain conditions.

To obtain the initial conditions leading to the three trajectories, $d\theta_\alpha/dh$ can be used.

For a-trajectory, condition $\frac{d\theta_a}{dh} \leq 0$ must be met, so that, Equ.20 can be obtain:

$$\frac{6h[\frac{h}{d+r}-\cos(\theta_0)]}{(d+r)\left\{1-\left[\frac{h}{d+r}-\cos(\theta_0)\right]^2\right\}} \geq -1 \qquad \text{(Equ.20)}$$

Since Equ.20 is a concave up function, so $d^2\theta_\alpha/dh^2$ can be used to determine the position of minimum vertex of Equ.20. After calculating, there is only one positive root of $d^2\theta_\alpha/dh^2$, $h=(d+r)\tan(\theta_0)*[1-\sin(\theta_0)]$. Then substitute this solution into Equ.20:

$$\frac{-6\sin(\theta_0)\left[\frac{\sin(\theta_0)-1}{\cos(\theta_0)}\right]^2}{1-\left[\frac{\sin(\theta_0)-1}{\cos(\theta_0)}\right]^2} \geq -1 \qquad \text{(Equ.21)}$$

It can be known that from Equ.21, a-trajectory is determined only by θ_0.

Furthermore, Equ.22, a part of $d\theta_\alpha/dh$, must be satisfied to prevent c-trajectory:

$$\frac{\psi^2 l^2 (r+0.5l)^2 2h}{g(d+r)^2\left\{1-\left[\frac{h}{d+r}-\cos(\theta_0)\right]^2\right\}^3} < 1 \qquad \text{(Equ.22)}$$

Differentiating Equ.22 with respect to h, to obtain the vertex position of Equ.22, and then substitute it into Equ.22:

$$\frac{\psi^2 l^2 (r+0.5l)^2 * \frac{2}{5}\left[2\cos(\theta_0) - \sqrt{\frac{9}{2}\cos(2\theta_0) - \frac{1}{2}}\right]}{g(d+r) * \left\{1 - \left[\frac{2\cos(\theta_0) - \sqrt{\frac{9}{2}\cos(2\theta_0) - \frac{1}{2}}}{5} - \cos(\theta_0)\right]^2\right\}^3} < 1 \quad \text{(Equ.23)}$$

It is evident that $\left[\frac{9}{2}\cos(2\theta_0) - \frac{1}{2}\right]$ must be equal or greater than 0, so that Equ.24 is obtained:

$$\theta_0 \geq \frac{1}{2} cos^{-1}\left(\frac{1}{9}\right) \quad \text{(Equ.24)}$$

And if $\left[\frac{1}{2} cos^{-1}\left(\frac{1}{9}\right)\right]$ substituted into Equ.21, the equation will be -1, which means the a-trajectory will occur if Equ.24 isn't satisfied.

Discussion

The above analysis only uses V_α, but it seems that V_β can also prevent the top from falling in some degree due to the gravity, because V_β is opposite to the direction of gravity. However, based on the analysis above, nutation is caused by precession taking away the kinetic energy of the falling top. And the resultant velocity of the gyroscope's linear velocity and V_β will only appear in the vertical plane parallel to the direction of gravity, while when viewed on the horizontal plane, the resultant velocity is consistent with the linear velocity. That is to say V_β would not transfer kinetic energy to the horizontal plane and has no effect on precession, so V_β would not affect the motion of the top.

Reference Used:

Methods of Analytical Dynamics by Leonard Meirovitch